Solutions of n-simplex Equation from Solutions of Braid Group Representation [1]

You-Quan Li[a] Zhan-Ning Hu[b, 2]

CCAST(World Lab.), P.O.Box 8730, Beijing 100080

[a] Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027

[b] Institute of Theoretical Physics, Academia Sinica,

P. O. Box 2735, Beijing 100080, China [3]

Abstract

It is shown that a kind of solutions of n-simplex equation can be obtained from representations of braid group. The symmetries in its solution space are also discussed.

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[2]**email address: huzn@itp.ac.cn**

[3]**mail address**

Recently many interests have been paid on the investigations of the higher dimensional integrable systems in the quantum field theory [1] and in the statistical mechanics [2]. For the lower dimension case of them, the Yang-Baxter equation (YBE) plays a crucial role of which the structure is now fairly well understood. As a substitution of YBE the tetrahedron equation becomes a integrability condition of the exactly solved model in three dimensions [3], from which the community of the layer-to-layer transfermatrixes is preserved. One of the approaches is the n-simplex equation [4] and it is said that the case of $n = 3$ is corresponding to the tetrahedron equation. The aim of this letter is to expose some procedure for deriving solutions of n-simplex equation from braid group representations (ie. solutions of parameter independent YBE) [5]. Meanwhile we would like to derive some symmetry transformation in solution space of 3-simplex equation as an example.

The 3-simplex equation we will consider takes the following form

$$R_{123}R_{214}R_{341}R_{432} = R_{234}R_{143}R_{412}R_{321} \tag{1}$$

where the order of subscripts are chosen in such a way that the normal of each surface of the 3-simplex is always toward the inside of the 3-simplex (tetrahedron)

Certainly, the positive direction of the normal of a surface determined by a cycle (for example, (123), (341) etc.) following the right-hand helicity. The matrices in eq.(1) stands for the scattering of three strings, for example

$$R_{214}|\mu_1,\mu_2,\mu_3,\mu_4 >= \sum_{\nu_1\nu_2\nu_3} R^{\nu_2\nu_1\nu_4}_{\mu_2\mu_1\mu_4}|\nu_1,\nu_2,\mu_3,\nu_4 > . \tag{2}$$

Solving solutions of eq.(1) is a complicated problem. It is known that many representations of braid group have been found in recently years. We will

show that if one have a representation of braid group, one can obtain a kind of solutions of the 3-simplex equation. A braid group is a category of free group under the constraint of the following equivalence relations

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}$$

$$b_i b_j = b_j b_i \quad for \quad |i-j| > 1. \tag{3}$$

It is called a braid group due to it has a simple realization on N-strings by identify

Then the equivalence relation (3) becomes an evident topological equivalence relation. If a representation of braid group takes as

$$\rho : b_i \longrightarrow S_{i,i+1} = I^{(1)} \otimes \cdots I^{(i-1)} \otimes S \otimes I^{(i+2)} \otimes \cdots I^{(N)} \tag{4}$$

where $S \in End(V \otimes V)$ satisfying the following parameter independent Yang-Baxter equation

$$S_{12} S_{23} S_{12} = S_{23} S_{12} S_{23}. \tag{5}$$

If we define an operator

$$t := \prod_{i=1}^{n} \prod_{j=1}^{i-1} b_i$$

which is understood as an ordered product from right to left or vise versa. We can show that the following identity holds

$$t_1 t_2 t_1 t_2 \cdots = t_2 t_1 t_2 t_1 \cdots, \tag{6}$$

where the number of t's in alternative product is $n+1$. The case $n=2$ is exactly the elementary equivalence relations of braid group eq.(3). For $n=3$ we have

$$t_1 t_2 t_1 t_2 = t_2 t_1 t_2 t_1, \tag{7}$$

where $t_1 = b_1 b_2 b_1, \quad t_2 = b_2 b_3 b_2$. Thus if we know a representation of braid group, we will have a solution of the following equation

$$\check{R}_{123}\check{R}_{234}\check{R}_{123}\check{R}_{234} = \check{R}_{234}\check{R}_{123}\check{R}_{234}\check{R}_{123} \tag{8}$$

where $\check{R}_{123} := \check{R} \otimes I, \check{R}_{234} := I \otimes \check{R}$ and $\check{R} \in End(V \otimes V \otimes V)$. This is easily realized by

$$\rho : t_1 \to \check{R}_{123}$$

due to $t_1 = b_1 b_2 b_1$ etc., then the following identities holds

$$\check{R}_{123} = S_{12}S_{23}S_{12} \otimes I \quad etc. \tag{9}$$

As to eq.(8), one may find some symmetry transformation of it. If one write out eq.(8) into component form instead of matrix form, one can easily find that the equation can be symbolized by Kauffman diagram. That says if we denote

$$\check{R}^{abc}_{def} \qquad , \quad \check{R}^{-1}{}^{abc}_{def}$$

The inverse relation and eq.(8) are depicted respectively as

and

$$(10)$$

where the inner line connecting legs of two shadows implies the summation over the repeated labels on the legs, and a simple vertical line stands for a unit matrix. It is not difficult to find that the diagram eq.(10) has the following symmetries:

Flipping via a horizontal axis, denoted by H

$$\tag{11}$$

or flipping via a vertical axis denoted by V

$$\tag{12}$$

or via both in term $VH = HV$.

$$\tag{13}$$

Thus we have

$$\begin{array}{ccc} (7) & \stackrel{H}{\rightarrow} & (8) \\ V\downarrow & & \downarrow V \\ (9) & \stackrel{H}{\rightarrow} & (10) \end{array}$$

and $H^2 = id, V^2 = id, HV = VH$. All the four diagrams eq.(10), eq.(11), eq.(12), eq.(13) depict the same equation eq.(8). So the solution space of eq.(8) has a discrete group symmetry. $\{id, H, V, VH | H^2 = id, V^2 = id, HV = VH\}$. The action of this group brings one solution of) into other three new solutions. i.e. if $\check{R}^{abc}_{def}$ is a solution of eq.(8), then $\check{R}'^{abc}_{def} = \check{R}^{cba}_{fed}$, $\check{R}''^{abc}_{def} = \check{R}^{edf}_{abc}$ and $\check{R}'''^{abc}_{def} = \check{R}^{fed}_{cba}$ will be solutions of eq.(8).

Furthermore, if giving a direction to the Kauffnman diagram $\check{R}^{abc}_{def}$, and adding a minus sign to the labels on the tip of the arrow, we can find that the summation of such labels on both side of the diagram eq.(10) are equal. This brings about a contineous transformation from a solution of eq.(8) into another

$$\check{R}^{abc}_{def} \rightarrow \check{R}'^{abc}_{def} = t^{a+b+c-d-e-f} \check{R}^{abc}_{def}. \tag{14}$$

Starting from the matrix form of eq.(8), we can obtain two more contineous transformations in solution space. They are an overall factor transformation $\check{R} \rightarrow \tau \check{R}$; a similar transformation by a tensor product of matrices $\check{R} \rightarrow (\Lambda \otimes \Lambda \otimes \Lambda) \check{R} (\Lambda^{-1} \otimes \Lambda^{-1} \otimes \Lambda^{-1})$. Because eigenvalues of a matrix are invariant under a similar transformation, the latter is a transformation within the subset of solution space, which is specialfied by the eigenvalues of $\check{R}$.

In above we made much discussion on eq.(8), now we introduce a new R-matrix

$$\check{R} = RP \tag{15}$$

where P is defined as

$$P|\mu_1, \mu_2, \mu_3 > := |\mu_3, \mu_2, \mu_1 > .$$

Then we can show the R-matrix satisfing the following equation as long as the $\check{R}$-matrix satisfing eq.(8)

$$R_{123} R_{214} R_{341} R_{432} = R_{234} R_{143} R_{412} R_{321} \tag{16}$$

which is an variant of the FM 3-simplex equation we have introduced at the begining of our discussion.

In a similar way, one may discuss the case of 4-simplex equation and so on. The key point is that eq.(6) is an identity on braid group, then if one has a representation of braid group, one can write down a expression from the expression of t_i on the basis of S-matrix, which is supposed to be solutions of parameter independent Yang-Baxter equation.

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